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Math 10/11 Enriched: Section 7.1 Graphing Circles and Ellipses

1. For each of the following equations, if it is a circle, find the coordinates of the center and the length of the radius. If it is an ellipse, find the center, lengths of the major and minor axis, and also the coordinates of the foci

a) $x^2+12x+y^2-12y=10$

$(x+6)^2+(y-6)^2=82$

$(h,k)=(-6,6)$

$r=\sqrt{82}$

b) $2x^2-16x+3y^2-24y=32$

$2(x-4)^2+3(y-4)^2=112$

$\frac{(x-4)^2}{56}+\frac{(y-4)^2}{\frac{112}{3}}=1$

$a=\sqrt{56}$

$b=\sqrt{\frac{112}{3}}=\frac{16\sqrt{21}}{3}$

$(h,k)=(4,4)$

$F_1=(h+\sqrt{a^2-b^2},k)=(4+\frac{2\sqrt{14}}{3},4)$

$F_2=(4-\frac{2\sqrt{14}}{3},14)$

c) $3x^2+15x-1+4y^2+28y+2=20$

$3(x+\frac{5}{2})^2+4(y+\frac{7}{2})^2=20+\frac{267}{4}=\frac{347}{4}$

$\frac{(x+\frac{5}{2})^2}{\frac{347}{12}}+\frac{(y+\frac{7}{2})^2}{\frac{347}{16}}=1$

$a=\sqrt{\frac{347}{12}}$ $b=\sqrt{\frac{347}{16}}$

$(h,k)=(-\frac{5}{2},-\frac{7}{2})$

$F=(-\frac{5}{2}\pm 2.689,-\frac{7}{2})$

d) $5x^2+25x+3+7y^2-14y-1=50$

$5(x+\frac{5}{2})^2+7(y-1)^2=\frac{345}{4}$

$\frac{(x+\frac{5}{2})^2}{\frac{69}{4}}+\frac{(y-1)^2}{\frac{345}{28}}=1$

$a=\frac{\sqrt{69}}{2}$

$(h,k)=(-\frac{5}{2},1)$

$F=(\frac{5}{2}\pm 2.22,1)$

e) $3x^2+20x+3y^2-19y=22$

$3(x+\frac{10}{3})^2+3(y-\frac{19}{6})^2=\frac{527}{3}$

$(x+\frac{10}{3})^2+(y-\frac{19}{3})^2=\frac{527}{9}$

$(h,k)=(-\frac{10}{3},\frac{19}{3})$

$r=\frac{\sqrt{527}}{3}$

12

f) $6x^2+10x+3y^2-8y=12$

$6(x-\frac{5}{6})^2+3(y-\frac{4}{3})^2=\frac{43}{2}$

$\frac{(x+\frac{5}{6})^2}{\frac{43}{12}}+\frac{(y-\frac{4}{3})^2}{\frac{43}{6}}=1$

$(h,k)=(-\frac{5}{6},\frac{4}{3})$

$a=\frac{43}{2\sqrt{3}}$ $b=\sqrt{\frac{43}{6}}$ $f=(-\frac{5}{6},\frac{4}{3}\pm 1.89)$

g) $8+x^2+15x=20-y^2+7y$

$(x+\frac{15}{2})^2+(y-\frac{7}{2})^2=\frac{161}{2}$

$(h,k)=(-\frac{15}{2},\frac{7}{2})$

$r=\sqrt{\frac{161}{2}}$

h) $3x^2=3(5-y)(5+y)+9(x+y)$

$3x^2=3(25-y^2)+9x+9y$

$x^2+y^2-3x-3y=25$

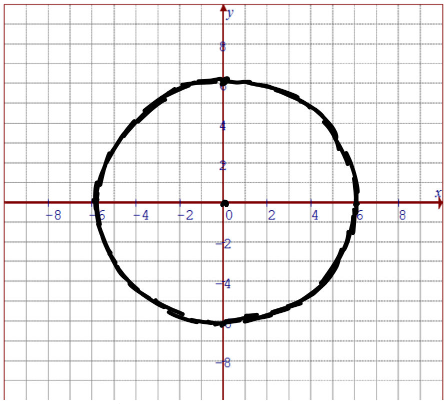
$(x-\frac{3}{2})^2+(y-\frac{3}{2})^2=\frac{59}{2}$

$(h,k)=(\frac{3}{2},\frac{3}{2})$

$r=\sqrt{\frac{59}{2}}=\frac{\sqrt{118}}{2}$

2. Given each equation below, graph it on the grid provided:

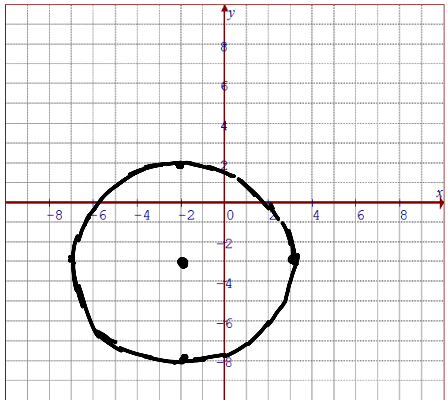
a) $x^2 + y^2 = 36$



Center: 0,0 Area: 36π

Domain: $-6 \le x \le 6$ Range: $-6 \le y \le 6$

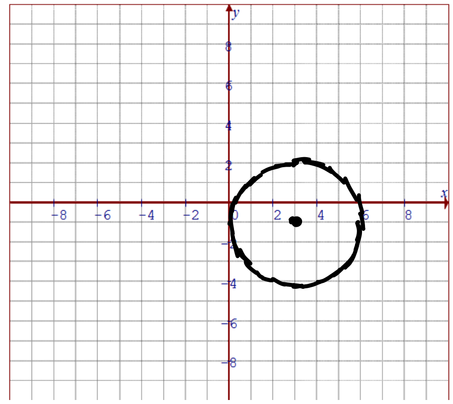
b) $(x+2)^2 + (y+2)^2 = 25$



Center: $-2,-2$ Area: 25π

Domain: $-7 \le x \le 3$ Range: $-8 \le y \le 2$

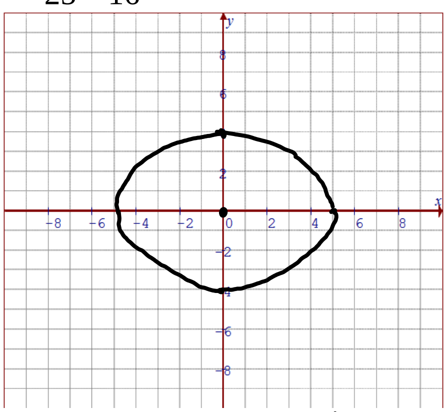
c) $(x-3)^2 + (y+1)^2 = 9$



Center: 3,-1 Area: 9π

Domain: $0 \le x \le 6$ Range: $-4 \le y \le 2$

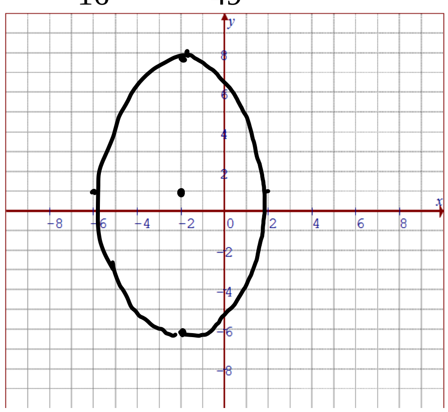
d) $\frac{x^2}{25} + \frac{y^2}{16} = 1$



Center: 0,0 Area: $\pi ab = 20\pi$

Domain: $-5 \le x \le 5$ Range: $-4 \le y \le 4$

e) $\frac{(x+2)^2}{16} + \frac{(y-1)^2}{49} = 1$

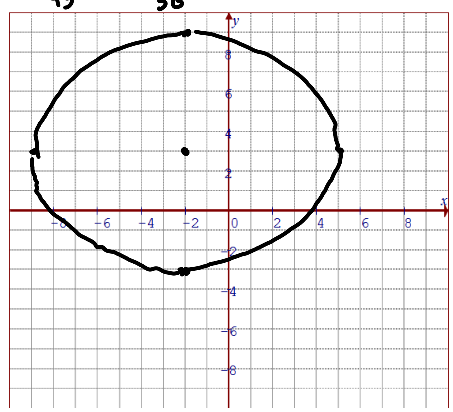


Center: $-2,1$ Area: $\pi ab = 28\pi$

Domain: $-6 \le x \le 2$ Range: $-6 \le y \le 8$

f) $36(x+2)^2 + 49(y-3)^2 = 1764$

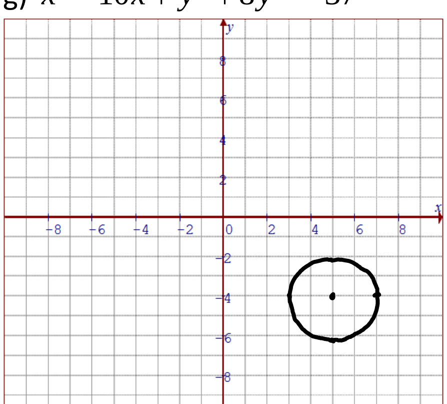
$\frac{(x+2)^2}{49} + \frac{(y-3)^2}{36} = 1$



Center: $-2,3$ Area: 42π

Domain: $-9 \le x \le 5$ Range: $-3 \le y \le 9$

g) $x^2 - 10x + y^2 + 8y = -37$

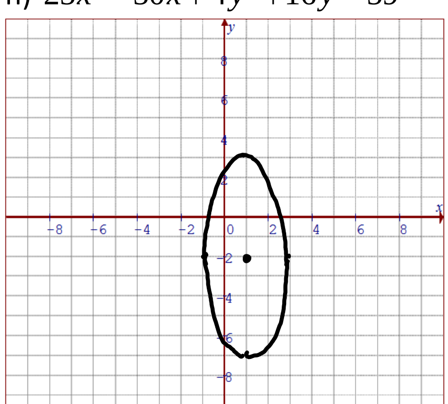


Center: $(5,-4)$ Area: 4π

Domain: $3 \le x \le 7$ Range: $-6 \le y \le -2$

$(x-5)^2 + (y+4)^2 = 4$

h) $25x^2 - 50x + 4y^2 + 16y = 59$

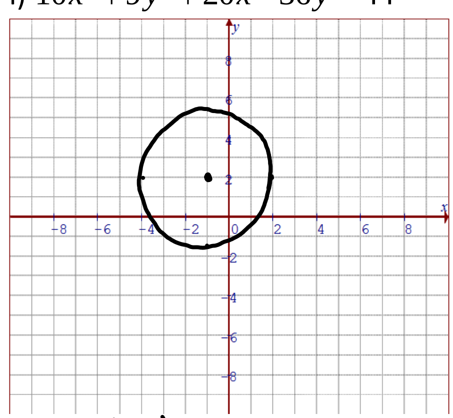


Center: $(1,-2)$ Area: 10π

Domain: $-1 \le x \le 3$ Range: $-7 \le y \le 3$

$25(x-1)^2 + 4(y+2)^2 = 100 \Rightarrow \frac{(x-1)^2}{4} + \frac{(y+2)^2}{25} = 1$

i) $10x^2 + 9y^2 + 20x - 36y = 44$



Center: $(-1,2)$ Area: $3\sqrt{10}\pi$

Domain: $-4 \le x \le 2$ Range: $2-\sqrt{10} \le y \le 2+\sqrt{10}$

$10(x+1)^2 + 9(y-2)^2 = 90$

$\frac{(x+1)^2}{9} + \frac{(y+2)^2}{10} = 1$

3. Given that the endpoints of a diameter are $(-3,11)$ and $(8,-7)$, what is the equation of the circle?

① $r = \frac{\sqrt{(11+7)^2+(-3-8)^2}}{2} = \frac{\sqrt{445}}{2}$

② $(h,k) = \text{midpoint} = \left(\frac{5}{2}, 2\right)$

$\Rightarrow \left(x-\frac{5}{2}\right)^2 + (y-2)^2 = \frac{445}{4}$

4. Given the equation of the ellipse $3x^2 - 8x + 2y^2 + 10y = 20$, convert it to the form of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$3\left(x-\frac{4}{3}\right)^2 + 2\left(y+\frac{5}{2}\right)^2 = \frac{227}{6} \Rightarrow \frac{\left(x-\frac{4}{3}\right)^2}{\frac{227}{18}} + \frac{\left(y+\frac{5}{2}\right)^2}{\frac{227}{12}} = 1$

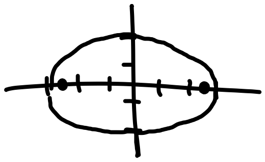
5. Which of the following has a greater area and by how much? $(x+2)^2 + (y-3)^2 = 25$ ① or $\frac{x^2}{25} + \frac{y^2}{36} = 1$ ②?

Area ①: $\pi r^2 = 25\pi$

Area ②: $\pi ab = 5 \cdot 6\pi = 30\pi$

② greater by 5π

6. If "P" is any point on the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$, "Q" has coordinates $(\sqrt{5},0)$ and "S" has coordinates $(-\sqrt{5},0)$, then find $PS + PQ$.



$F_1, F_2 = (\pm\sqrt{5}, 0)$

$F_1P + F_2P$ is constant.

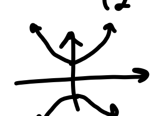
let $P = (3,0)$: $F_1P + F_2P = 6$

7. Find the distance between the foci of the conic section whose equation is: $4x^2 + 8x + 13 = 3y^2 - 18y$

$4(x+1)^2 - 3(y-3)^2 = -13-23$

$-4(x+1)^2 + 3(y-3)^2 = 36$

$-\frac{(x+1)^2}{9} + \frac{(y-3)^2}{12} = 1$



$F = (h, k \pm \sqrt{a^2+b^2})$

$F = (-1, 3 \pm \sqrt{21})$

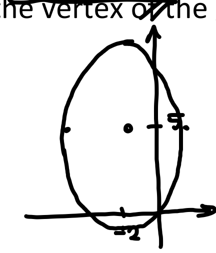
Interfoci length $= 2\sqrt{21}$

8. For what values of "k" will the following equation be an ellipse? $3x^2 + 15xy + ky^2 = 1025$

$(\sqrt{3}x + \sqrt{k}y)^2 = 1025 \Rightarrow 2\sqrt{3k}\,xy > 15xy \Rightarrow 3k \geq \frac{225}{4} \Rightarrow k > \frac{225}{12}$

$\frac{225}{12} < k$ ← if $k = \frac{225}{4}$, you'll have parallel lines.

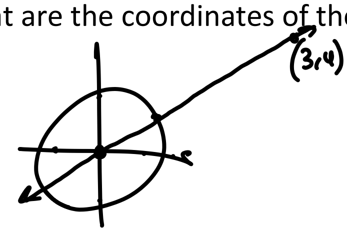
9. What are the coordinates, in the form of (x,y) of the vertex of the given conic section that is farthest from the origin? $\frac{(x+2)^2}{16} + \frac{(y-5)^2}{36} = 1$



$b = 6$ vertex $= (h, k+b)$

$= (-2, 11)$

10. What are the coordinates of the point on the graph of $x^2 + y^2 = 1$ that is nearest to (3,4)?



slope $= \frac{4}{3}$ y-int $= 0$

$y = \frac{4}{3}x$, $x^2 + y^2 = 1$ → $x^2 + \frac{16}{9}x^2 = 1 \Rightarrow x = \frac{3}{5}$

$y = \frac{4}{5}$

$(x,y) = \left(\frac{3}{5}, \frac{4}{5}\right)$

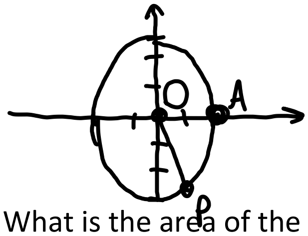
11. The circle defined by the equation $(x+4)^2+(y-3)^2=9$ is moved horizontally until its centre is on the line $x=6$. How far does the centre of the circle move?

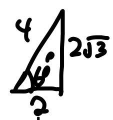
$(h,k)=(-4,3)$

H.S. 10R

10 units

12. What are the coordinates of the point "P" on the lower half of the graph of the ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1$ for which $\angle AOP$, determined by points A(2,0), O(0,0), and P, has a measure of $60°$?





slope $OP = \frac{-2\sqrt{3}}{2} = -\sqrt{3}$

$y=-\sqrt{3}x$ and $\frac{x^2}{4}+\frac{y^2}{9}=1$ intersect at $\left(2\sqrt{\frac{3}{7}}, \frac{-6}{\sqrt{7}}\right)$

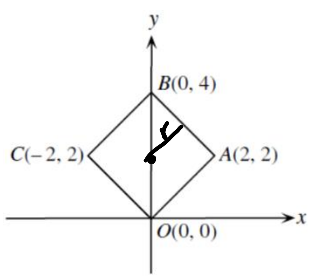
13. What is the area of the shape bounded by the curve: $4x^2-8x+y^2+4y=0$?

$4(x-1)^2+(y+2)^2=8$

$\frac{(x-1)^2}{2}+\frac{(y+2)^2}{8}=1$

Area $= ab\pi = \sqrt{2}\cdot\sqrt{8}\cdot\pi = 4\pi$

14. Square OABC is drawn with vertices as shown. Find the equation of the circle with the largest area that can be drawn inside the square. Euclid



$\overline{AO}=2\sqrt{2}$

$r=\sqrt{2}$

$(h,k)=(0,2)$

$x^2+(y-2)^2=2$

15. What is the area of the ellipse defined by $\frac{(x+3)^2}{16}+\frac{(y-4)^2}{25}=1$

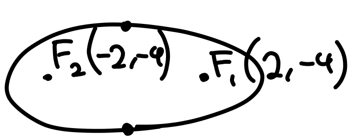
$ab\pi = 4\cdot5\cdot\pi = 20\pi$

16. A point moves so that the sum of its distances from $(-2,-4)$ and $(2,-4)$ is 16. Find the coordinates of the endpoint of the minor axis that is below the major axis.

$\sqrt{a^2-b^2}=2$

$2a=16$

$a=8$

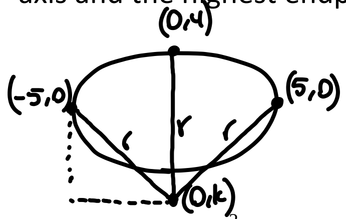


$\sqrt{8^2-b^2}=2 \Rightarrow b^2=60$

$b=\sqrt{60}=4\sqrt{15}$

coordinates $=(0,-4-2\sqrt{15})$

17. Consider the graph of $\frac{x^2}{25}+\frac{y^2}{16}=1$. A circle is graphed which contains the endpoints of the major axis and the highest endpoint of the minor axis. Find the center of the circle.



① $r = 4 - k$

② $5^2 + k^2 = (4-k)^2$

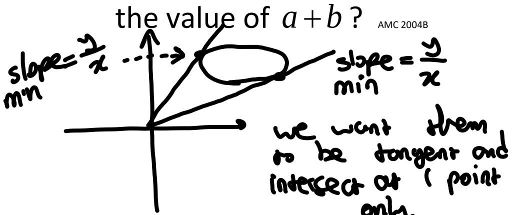
$25 + k^2 = 16 - 8k + k^2$

$8k = -9$

$k = -\frac{9}{8}$

$(h,k) = (0, -\frac{9}{8})$

18. The graph of $2x^2 + xy + 3y^2 - 11x - 20y + 40 = 0$ is an ellipse in the first quadrant of the xy-plane. Let "a" and "b" be the maximum and minimum values of $\frac{y}{x}$ over all points (x,y) on the ellipse. What is the value of $a+b$? AMC 2004B



$y = m$ and $2x^2 + xy + 3y^2 - 11x - 20y + 40 = 0$

$2x^2 + xm + 3m^2 - 11x - 20m + 40 = 0$

Since tangent, only 1 solution of x should exist. → $2x^2 + x(m-11) + 3m^2 - 20m + 40$

$\Delta = \sqrt{(m-11)^2 - 4(2)(3m^2-20m+40)} = \sqrt{-23m^2+138m-199} = 0$

$23m^2 - 138m + 199 = 0$

$m_1 = 2.4$

$m_2 = 3.6$

$a+b = m_1 + m_2 = 5.99$

19. What is the greatest possible value of "*a*" for which there is at least one real solution (*x*,*y*) to the system $x^2 + y^2 = 1$ and $x^2y^2 = a$?

$x^2 = \frac{a}{y^2} \Rightarrow \frac{a}{y^2} + y^2 = 1 \Rightarrow a + y^4 - y^2 = 0$

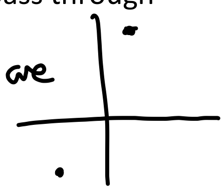
$y^4 - y^2 + a = 0$

$\Delta = \sqrt{1-4a}$ $\quad 1 - 4a \geq 0 \Rightarrow a \leq \frac{1}{4}$

$a = \frac{1}{4}$

20. In the coordinate plane, any circle which passes through (-2,-2) and (1,4) cannot also pass through (x,2006). What is the value of "x"?

the 3 points can NEVER form a straight line if they are on the same circle.



$m = \frac{6}{3} = 2 \Rightarrow \frac{2006-4}{x-1} = 2 \Rightarrow x = 1002$

21. Challenge: The circles $(x-p)^2 + y^2 = r^2$ has centre "C" and circle $x^2 + (y-p)^2 = r^2$ has centre "D". The circles intersect at two distinct points "A" and "B", with x-coordinates "a" and "b", respectively. Euclid

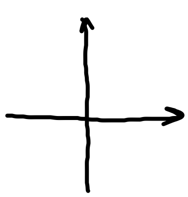
a. Prove that $a + b = p$ and $a^2 + b^2 = r^2$

$C(h,k) = (p,0)$ $\quad D(h,k) = (0,p)$

$(x-p)^2 + y^2 = x^2 + (y-p)^2$

$x^2 - 2px + p^2 + y^2 = x^2 + y^2 - 2py + p^2 \Rightarrow x = y \Rightarrow a = b = \frac{p}{2}$ since halfway through $\Rightarrow a + b = p$

b. If "r" is fixed and "p" is then found to maximize the area of quadrilateral CABD, prove that either "A" or "B" is the origin.



c. If "p" and "r" are integers, determine the minimum possible distance between "A" and "B". Find positive integers "p" and "r", each larger than 1, that give this distance.

Challenge problems:

[Aime 2022 12]

Let $a, b, x,$ and y be real numbers with $a > 4$ and $b > 1$ such that

$$\frac{x^2}{a^2}+\frac{y^2}{a^2-16}=\frac{(x-20)^2}{b^2-1}+\frac{(y-11)^2}{b^2}=1.$$

Find the least possible value of $a+b$.

What are the coordinates of the point on the graph of $x^2+y^2=1$ that is nearest to (3,4)?

The graph of $2x^2+xy+3y^2-11x-20y+40=0$ is an ellipse in the first quadrant of the xy-plane. Let a and b be the maximum and minimum values of $\frac{y}{x}$ over all points (x, y) on the ellipse. What is the value of $a+b$?

(A) 3 **(B)** $\sqrt{10}$ **(C)** $\frac{7}{2}$ **(D)** $\frac{9}{2}$ **(E)** $2\sqrt{14}$

The points $A(-8,6)$ and $B(-6,-8)$ lie on the circle $x^2+y^2=100$.

(a) Determine the equation of the line through A and B.

(b) Determine the equation of the perpendicular bisector of AB.

(c) The perpendicular bisector of AB cuts the circle at two points, P in the first quadrant and Q in the third quadrant. Determine the coordinates of P and Q.

(d) What is the length of PQ? Justify your answer.

A circle having center $(0,k)$, with $k>6$, is tangent to the lines $y=x$, $y=-x$ and $y=6$. What is the radius of this circle?

(A) $6\sqrt{2}-6$ **(B)** 6 **(C)** $6\sqrt{2}$ **(D)** 12 **(E)** $6+6\sqrt{2}$

Let C_1 and C_2 be circles defined by

$$(x-10)^2+y^2=36$$

and

$$(x+15)^2+y^2=81,$$

respectively. What is the length of the shortest line segment $\overline{PQ}$ that is tangent to C_1 at P and to C_2 at Q?

(A) 15 **(B)** 18 **(C)** 20 **(D)** 21 **(E)** 24

Problem 4. Find a point (u, v) on the ellipse with equation $x^2 + 2y^2 = 1$ such that u and v are rational, and each, when expressed as a reduced fraction, has a denominator greater than 1000. Hint: Consider the line with slope m that passes through the point $(-1, 0)$.

Substitute $(x+1)m$ for y in $x^2 + 2y^2 = 1$. After a little simplification, we obtain

$$(1 + 2m^2)x^2 + 4m^2x + 2m^2 - 1 = 0.$$

Now we could use the Quadratic Formula, or even, unusually, factorization, to solve for x. But this is not necessary. For the product of the roots is $(2m^2 - 1)/(1 + 2m^2)$, and -1 is one of the roots, so the other root is given by $u = (1 - 2m^2)/(1 + 2m^2)$. The corresponding v is given by $v = 2m/(1 + 2m^2)$.

Note that if m is rational, then u and v are rational. (Parenthetically, if (u, v) lies on the ellipse, with u and v rational, with $u \neq -1$, then the slope of the line joining $(-1, 0)$ to (u, v) is rational. So all rational points (u, v) on the ellipse except for $(-1, 0)$ can be obtained through this process with m rational.)

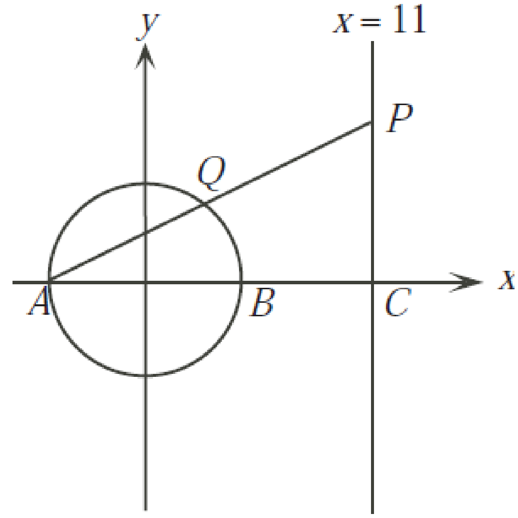
Now everything is easy. Take for example $m = 100$. That gives $u = 19999/20001$ and $v = 200/20001$.

Comment. More interestingly, the same process can be used with the circle $x^2 + y^2 = 1$. We find that apart from $(-1, 0)$, all the rational points (u, v) on the unit circle are given by $u = (1 - m^2)/(1 + m^2)$, $v = 2m/(1 + m^2)$, where m ranges over the rationals.

10. A point moves so that the sum of its distances from $(-2, -4)$ and $(2, 4)$ is 16. Find the coordinates of the endpoint of the minor axis that is below the major axis. $\left(\frac{4\sqrt{55}}{5}, -\frac{2\sqrt{55}}{5}\right)$

17. Consider the graph of $\dfrac{x^2}{25} + \dfrac{y^2}{16} = 1$. A circle is graphed which contains the endpoints of the major axis and the highest endpoint of the minor axis. Find the center of this circle. $(0, -\frac{9}{8})$

2. In the diagram, the circle $x^2 + y^2 = 25$ intersects the x-axis at points A and B. The line $x = 11$ intersects the x-axis at point C. Point P moves along the line $x = 11$ above the x-axis and AP intersects the circle at Q.



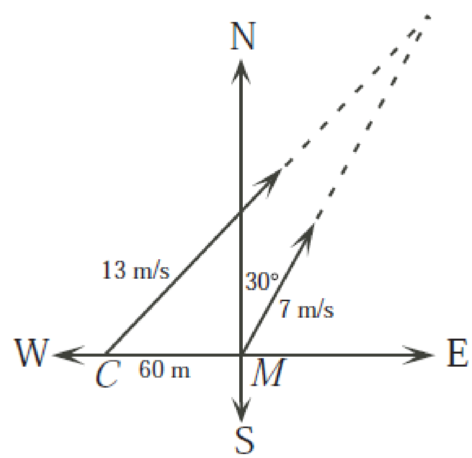
(a) Determine the coordinates of P when $\triangle AQB$ has maximum area. Justify your answer.

(b) Determine the coordinates of P when Q is the midpoint of AP. Justify your answer.

(c) Determine the coordinates of P when the area of $\triangle AQB$ is $\frac{1}{4}$ of the area of $\triangle APC$. Justify your answer.

Challenge COMC #4

4. A cat is located at C, 60 metres directly west of a mouse located at M. The mouse is trying to escape by running at 7 m/s in a direction 30° east of north. The cat, an expert in geometry, runs at 13 m/s in a suitable straight line path that will intercept the mouse as quickly as possible.



(a) If t is the length of time, in seconds, that it takes the cat to catch the mouse, determine the value of t.

(b) Suppose that the mouse instead chooses a different direction to try to escape. Show that no matter which direction it runs, all points of interception lie on a circle.

(c) Suppose that the mouse is intercepted after running a distance of d_1 metres in a particular direction. If the mouse would have been intercepted after it had run a distance of d_2 metres in the opposite direction, show that $d_1 + d_2 \geq 14\sqrt{30}$.

2. (a) The circle defined by the equation $(x+4)^2+(y-3)^2=9$ is moved horizontally until its centre is on the line $x=6$. How far does the centre of the circle move?

1. The points $A(-8,6)$ and $B(-6,-8)$ lie on the circle $x^2+y^2=100$.

   (a) Determine the equation of the line through A and B.

   (b) Determine the equation of the perpendicular bisector of AB.

   (c) The perpendicular bisector of AB cuts the circle at two points, P in the first quadrant and Q in the third quadrant. Determine the coordinates of P and Q.

   (d) What is the length of PQ? Justify your answer.

9. The circle $(x-p)^2+y^2=r^2$ has centre C and the circle $x^2+(y-p)^2=r^2$ has centre D. The circles intersect at two *distinct* points A and B, with x-coordinates a and b, respectively.

   (a) Prove that $a+b=p$ and $a^2+b^2=r^2$.

   (b) If r is fixed and p is then found to maximize the area of quadrilateral $CADB$, prove that either A or B is the origin.

   (c) If p and r are integers, determine the minimum possible distance between A and B. Find positive integers p and r, each larger than 1, that give this distance.

7. Each of the points $P(4,1)$, $Q(7,-8)$ and $R(10,1)$ is the midpoint of a radius of the circle C. Determine the length of the radius of circle C.

What are the coordinates, in the form (x,y), of the vertex of the given conic section that is farthest from the origin? $\frac{(x+2)^2}{16}+\frac{(y-5)^2}{36}=1$

2. (a) The circle defined by the equation $(x+4)^2+(y-3)^2=9$ is moved horizontally until its centre is on the line $x=6$. How far does the centre of the circle move?

What is the area of the shape bounded by the curve: $4x^2-8x+y^2+4y=0$?

If P is any point on the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$, Q has coordinates $(\sqrt{5},0)$ and S has coordinates $(-\sqrt{5},0)$, then find $PQ+PS$. 6

Find the distance between the foci of the conic section whose equation is:

$$4x^2 + 8x + 13 = 3y^2 - 18y \qquad 2\sqrt{21}$$

6. For what values of k will the following eq determine an ellipse?

$$3x^2 + 15xy + ky^2 = 1025 \qquad k > \frac{75}{4}$$

10. A point moves so that the sum of its distai from $(-2,-4)$ and $(2,4)$ is 16. Find the coordinates of the endpoint of the minor a: that is below the major axis. $\left(\frac{4\sqrt{55}}{5}, -\frac{2\sqrt{55}}{5}\right)$